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Linear sigma models of H and KK monopoles

Kazumi Okuyama

Department of Physics and Astronomy, University of British Columbia
Vancouver, BC, V6T 1Z1, Canada
E-mail: kazumi@phas.ubc.ca

Abstract: We propose a gauged linear sigma model of k H-monopoles. We also consider the T-dual of this model describing KK-monopoles and clarify the meaning of "winding coordinate" studied recently in hep-th/0507204.

Keywords: Solitons Monopoles and Instantons, Sigma Models, p-branes.

Contents

1. Introduction

The T-duality between H-monopoles (NS5-branes on $\mathbf{S}^1$) [14−16] and KK-monopoles (multi-Taub-NUT space) [12, 13] is a well-established duality [17]. At the level of supergravity this duality exchanges the two gauge fields B_{i9} and g_{i9} where $i = 6, 7, 8$ and x^9 is the $\mathbf{S}^1$ direction (we take the worldvolume of monopoles to be extended in the 012345 directions).

However, it was pointed out [2] that to fully understand the correspondence of collective coordinates of these monopoles we have to take into account of the stringy effects. A single H-monopole carries collective coordinates $\mathbf{R}^3 \times \mathbf{S}^1$ which correspond to the transverse space of the NS5-brane.[1] On the other hand, the Taub-NUT space has a moduli $\mathbf{R}^3$ which specifies the degeneration locus of the $\mathbf{S}^1$ fiber and hence $\mathbf{S}^1$ part of the moduli on the H-monopole side is missing in the geometric picture of KK-monopoles. In other words, KK-monopoles are smeared in the $\mathbf{S}^1$ direction and there is no definite position in $\mathbf{S}^1$. It is known that the missing moduli comes from the zero-modes of the B-field given by the harmonic two-forms on the Taub-NUT space [5]. Since B-field couples to the winding number, the $\mathbf{S}^1$ moduli of KK-monopole is not a geometric translation mode but the shift symmetry in the "winding coordinate" [2].

[1] We consider NS5-branes in type-IIB theory. NS5-branes in type-IIA theory have extra $\mathbf{S}^1$ moduli [24, 25].

In a recent paper [1], it is argued that a gauged linear sigma model (GLSM) [18] is a useful tool to analyze this problem. In [1] the GLSM of a single KK-monopole was studied by taking the T-dual of the GLSM for a H-monopole obtained by Tong [3], and the worldsheet instanton corrections to the effective sigma-model metric is computed. This result seems puzzling from the effective nonlinear sigma model perspective since there is no closed two-cycle in a single KK-monopole background where the worldsheet is supposed to wrap.

In this paper, we generalize [3, 1] and construct the GLSM of multiple H-monopoles and its T-dual GLSM of the multiple KK-monopoles. We argue that the moduli of H- and KK-monopoles appear in GLSM as parameters of two-dimensional field theory and clarify the physical meaning of "winding moduli" of KK-monopoles by analyzing the effective theory obtained from the GLSM. We also suggest a possible resolution to the above-mentioned puzzle.

The organization of this paper is as follows. In section 2, we construct a GLSM for k H-monopoles and study its low energy behavior and instanton corrections. In section 3, by taking the T-duality of H-monopole GLSM we obtain the GLSM for k KK-monopoles. In section 4, we discuss the instanton effect of KK-monopole model from the effective theory perspective.

2. GLSM for H-monopoles

In this section, we construct a GLSM for the k H-monopoles by generalizing the known model for $k = 1$ [3]. Our model is a $\mathcal{N} = (4, 4)$ $\mathrm{U}(1)^k$ gauge theory with k charged hypermultiplets and one neutral twisted hypermultiplet.

2.1 The model

We use the $\mathcal{N} = 2$ superfield notation. We argue that k H-monopoles are described by the GLSM with the following D-term $\mathcal{L}_D$, the superpotential term $\mathcal{L}_F$, and the twisted superpotential term $\mathcal{L}_{\widetilde{F}}$:

$$
\begin{aligned}
\mathcal{L}_D &= \int d^4\theta \frac{1}{g^2}(-\Theta^\dagger\Theta + \Psi^\dagger\Psi) + \sum_{a=1}^{k}\left\{\frac{1}{e_a^2}(-\Sigma_a^\dagger\Sigma_a + \Phi_a^\dagger\Phi_a) + Q_a^\dagger e^{V_a}Q_a + \widetilde{Q}_a^\dagger e^{-V_a}\widetilde{Q}_a\right\} \\
\mathcal{L}_F &= \int d\theta^+ d\theta^- \sum_{a=1}^{k}\left\{\widetilde{Q}_a\Phi_a Q_a + (s_a - \Psi)\Phi_a\right\}, \\
\mathcal{L}_{\widetilde{F}} &= \int d\theta^+ d\bar{\theta}^- \sum_{a=1}^{k}(t_a - \Theta)\Sigma_a\,. \qquad (2.1)
\end{aligned}
$$

Let us explain our notation in 2.1. (Φ_a, Σ_a) is the $\mathcal{N} = 4$ U(1) vector multiplet, $(Q_a, \widetilde{Q}_a)$ is the charged hypermultiplet, and (Ψ, Θ) is the neutral twisted hypermultiplet. We can see that 2.1 reduces to the model in [3] when we set $k = 1$. (s_a, t_a) in $\mathcal{L}_F$ and $\mathcal{L}_{\widetilde{F}}$ are some constant parameters corresponding to the FI-parameters and the theta-angle

$$
s_a = r_a^1 + ir_a^2\,, \qquad t_a = r_a^3 + i\theta_a\,. \qquad (2.2)
$$

In the following, we use the lower case letters for the scalar field component of the superfield except for the twisted hypermultiplet. Following [3], we denote the scalar component of twisted hypermultiplet as (r^1, r^2, r^3, θ):

$$\begin{aligned} \Psi &= r^1 + ir^2 + \sqrt{2}\theta^+\chi_+ + \sqrt{2}\theta^-\chi_- + \cdots \\ \Theta &= r^3 + i\theta - i\sqrt{2}\theta^+\overline{\widetilde{\chi}}_+ - i\sqrt{2}\bar{\theta}^-\widetilde{\chi}_- + \cdots . \end{aligned} \tag{2.3}$$

We also use the vector notation $\vec{r} = (r^1, r^2, r^3)$ to emphasize the triplet of R-symmetry $\mathrm{SU}(2)_R$. The important property of this lagrangian is that the scalar fields $(\vec{r}, \theta)$ act as the dynamical FI-parameters and the dynamical theta-angle.

It is easy to write down the component field expression of the lagrangian in the Wess-Zumino gauge. Here we focus on the bosonic part of the action

$$S = \frac{1}{2\pi}\int d^2x(\mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{pot}} + \mathcal{L}_{\mathrm{top}} + \mathcal{L}_{\mathrm{fermion}}) \tag{2.4}$$

where $\mathcal{L}_{\mathrm{kin}}$ is the kinetic term of bosonic fields

$$\mathcal{L}_{\mathrm{kin}} = -\frac{1}{2g^2}\Big((\partial\vec{r})^2 + (\partial\theta)^2\Big) + \sum_{a=1}^{k}\left\{\frac{1}{2e_a^2}\Big((F_{01}^a)^2 - |\partial\phi_a|^2 - |\partial\sigma_a|^2\Big) - |\mathcal{D}q_a|^2 - |\mathcal{D}\widetilde{q}_a|^2\right\} \tag{2.5}$$

and $\mathcal{L}_{\mathrm{pot}}$ is the potential term

$$\begin{aligned} \mathcal{L}_{\mathrm{pot}} = \sum_{a=1}^{k}\Big\{ &-\frac{e_a^2}{2}(|q_a|^2 - |\widetilde{q}_a|^2 - r^3 + r_a^3)^2 - \frac{e_a^2}{2}|2q_a\widetilde{q}_a - (r^1 + ir^2) + (r_a^1 + ir_a^2)|^2 - \\ &- (|\phi_a|^2 + |\sigma_a|^2)(|q_a|^2 + |\widetilde{q}_a|^2 + g^2)\Big\} \end{aligned} \tag{2.6}$$

and $\mathcal{L}_{\mathrm{top}}$ is the topological term

$$\mathcal{L}_{\mathrm{top}} = \sum_{a=1}^{k}(\theta - \theta_a)F_{01}^a . \tag{2.7}$$

Note that due to this axionic coupling the field θ becomes a compact boson $\theta \sim \theta + 2\pi$. Note also that the total space of parameters $\{(\vec{r}_a, \theta_a)\}_{a=1}^k$ is $(\mathbf{R}^3 \times \mathbf{S}^1)^k/S_k$. The quotient by S_k comes from the fact that the permutation of the k gauge groups leads to the same theory. We will see in section 2.3 that this parameter space has the spacetime interpretation as the moduli space of k H-monopoles.

2.2 The low energy theory

Now let us consider the effective theory of this model by restricting the lagrangian to the massless modes. From the expression of $\mathcal{L}_{\mathrm{pot}}$ 2.7, the moduli space of vacua[2] is characterized by the equations

$$F_{01}^a = \sigma_a = \phi_a = 0, \quad |q_a|^2 - |\widetilde{q}_a|^2 = r^3 - r_a^3 , \qquad 2q_a\widetilde{q}_a = r^1 + ir^2 - (r_a^1 + ir_a^2) . \tag{2.8}$$

[2]Strictly speaking, there is no moduli space of vacua in two dimensions because of the Coleman theorem [21]. We analyze the low energy theory in the spirit of Born-Oppenheimer approximation.

From the last two equations in 2.8, it follows that

$$|q_a|^2 + |\tilde{q}_a|^2 = |\vec{r} - \vec{r}_a| \,. \tag{2.9}$$

In the IR limit $e_a \to \infty$, the vector multiplet and charged hypermultiplets become massive and they can be integrated out. The crucial step is to rewrite the kinetic term of hypermultiplet restricted on the locus 2.8

$$|\mathcal{D}q_a|^2 + |\mathcal{D}\tilde{q}_a|^2 = \frac{1}{4|\vec{r} - \vec{r}_a|}(\partial\vec{r})^2 + \frac{|\vec{r} - \vec{r}_a|}{4}(2A^a + 2\partial\varphi_a + \vec{\omega}_a \cdot \partial\vec{r})^2 \tag{2.10}$$

where $\varphi_a = -\arg(iq_a)$ and $\vec{\omega}_a$ is given by

$$\nabla \times \vec{\omega}_a = \nabla \frac{1}{|\vec{r} - \vec{r}_a|} \,. \tag{2.11}$$

After setting $\frac{1}{2e_a^2}(F^a_{01})^2 \to 0$, the action becomes quadratic in A^a_μ. Therefore we can integrate out the gauge field classically and we find

$$A^a_\mu = -\partial_\mu\varphi_a - \frac{1}{2}\vec{\omega}_a \cdot \partial_\mu\vec{r} + \frac{1}{2|\vec{r} - \vec{r}_a|}\epsilon_{\mu\nu}\partial^\nu\theta \,. \tag{2.12}$$

Plugging this back into the original lagrangian, the effective lagrangian for the twisted hypermultiplet is found to be

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}H(\partial_\mu\vec{r} \cdot \partial^\mu\vec{r} + \partial_\mu\theta\partial^\mu\theta) + \epsilon_{\mu\nu}\vec{\omega} \cdot \partial^\mu\vec{r}\partial^\nu\theta \tag{2.13}$$

where

$$H = \frac{1}{g^2} + \frac{1}{2}\sum_{a=1}^{k}\frac{1}{|\vec{r} - \vec{r}_a|}\,, \qquad \vec{\omega} = \frac{1}{2}\sum_{a=1}^{k}\vec{\omega}_a\,, \qquad \nabla \times \vec{\omega} = \nabla H. \tag{2.14}$$

This lagrangian 2.13 is nothing but the nonlinear sigma model describing the k NS5-branes smeared in the $\mathbf{S}^1_\theta$ direction. The last term in 2.13 represents the coupling to the B-field generated by the NS5-branes. As discussed in [3], the localization in the θ direction is recovered by summing over the instanton effects, which we will turn next.

2.3 Worldsheet instanton corrections

The instanton effects in the $k = 1$ model is studied in [3]. In the case of k H-monopoles, we expect that the instanton effects in different U(1) sectors are decoupled from each other in the first approximation. Therefore, we can simply use the result of [3] for the single U(1) theory and sum over the k U(1) contributions. The instanton action in the a^{th} U(1) sector is given by [3]

$$S^{\text{inst}}_{n_a} = |n_a||\vec{r} - \vec{r}_a| - in_a(\theta - \theta_a) \tag{2.15}$$

where n_a is the instanton number in the a^{th} U(1) gauge field

$$n_a = -\int_\Sigma \frac{F^a}{2\pi}\,. \tag{2.16}$$

In general it is quite difficult to compute the n-instanton effect since it involves a complicated integral over the instanton moduli space. However, it is conjectured that all instanton numbers contribute with equal weight in the effective metric [3] to match the target space picture. Applying this conjecture to our case, we expect that H in Hom is corrected to

$$\begin{aligned} H &= \frac{1}{g^2} + \frac{1}{2}\sum_{a=1}^{k} \frac{1}{|\vec{r}-\vec{r_a}|} \sum_{n_a\in\mathbb{Z}} e^{-|n_a||\vec{r}-\vec{r_a}|+in_a(\theta-\theta_a)} \\ &= \frac{1}{g^2} + \frac{1}{2}\sum_{a=1}^{k} \frac{\sinh|\vec{r}-\vec{r_a}|}{|\vec{r}-\vec{r_a}|} \frac{1}{\cosh|\vec{r}-\vec{r_a}| - \cos(\theta-\theta_a)} \\ &= \frac{1}{g^2} + \sum_{a=1}^{k}\sum_{m_a\in\mathbb{Z}} \frac{1}{|\vec{r}-\vec{r_a}|^2 + (\theta-\theta_a+2\pi m_a)^2}\,. \end{aligned} \tag{2.17}$$

The sigma model 2.13 with this instanton corrected H 2.17 corresponds to the k NS5-branes localized at $(\vec{r_a}, \theta_a) \in \mathbf{R}^3 \times \mathbf{S}^1$. From the last expression in 2.17, we can see that this corresponds to the periodic array of CHS solution [23]. This effective theory preserves a $\mathcal{N} = (4,4)$ SUSY since it is a nonlinear sigma model of hyperKähler with torsion [10, 11]. As is clear from the derivation, the collective coordinates $(\vec{r_a}, \theta_a)$ of NS5-branes appear in the GLSM as the FI-parameter and the theta-parameter.

From the first expression of H in 2.17, we can see that the instanton number n_a has a spacetime interpretation as the momentum along $\mathbf{S}^1_\theta$ and the collective coordinates θ_a's are conjugate to this momentum mode. In other words, θ_a is the translation mode in the $\mathbf{S}^1_\theta$ direction.

3. GLSM for KK-monopoles

In this section, we consider the T-dual of H-monopole GLSM 2.1 following the general prescription [9]. We dualize the $\mathcal{N} = 2$ twisted chiral multiplet Θ as was done for the $k = 1$ case [1]. To do this, we first rewrite the F-term involving Θ as a D-term

$$\int d^4\theta - \frac{1}{g^2}\Theta^\dagger\Theta - (\Theta + \Theta^\dagger)\sum_{a=1}^{k} V_a\,. \tag{3.1}$$

Then we use the usual trick of introducing a real superfield B and a chiral superfield P

$$\int d^4\theta - \frac{1}{2g^2}B^2 - \frac{1}{g^2}\Big(P + P^\dagger + g^2\sum_{a=1}^{k} V_a\Big)B\,. \tag{3.2}$$

3.2 goes back to 3.1 by integrating out P, while integrating out B leads to

$$\int d^4\theta \frac{1}{2g^2}\Big(P + P^\dagger + g^2\sum_{a=1}^{k} V_a\Big)^2\,. \tag{3.3}$$

Therefore, the T-dual of 2.1 is given by

$$
\begin{aligned}
\mathcal{L}_D &= \int d^4\theta \frac{1}{g^2}\Psi^\dagger\Psi + \frac{1}{2g^2}\Big(P + P^\dagger + g^2\sum_{a=1}^{k} V_a\Big)^2 + \\
&\quad + \sum_{a=1}^{k}\left\{\frac{1}{e_a^2}(-\Sigma_a^\dagger\Sigma_a + \Phi_a^\dagger\Phi_a) + Q_a^\dagger e^{V_a} Q_a + \widetilde{Q}_a^\dagger e^{-V_a}\widetilde{Q}_a\right\} \\
\mathcal{L}_F &= \int d\theta^+ d\theta^- \sum_{a=1}^{k}\Big\{\widetilde{Q}_a\Phi_a Q_a + (s_a - \Psi)\Phi_a\Big\}, \qquad \mathcal{L}_{\widetilde{F}} = \int d\theta^+ d\bar{\theta}^- \sum_{a=1}^{k} t_a\Sigma_a\,. \qquad (3.4)
\end{aligned}
$$

Again it is easy to write down the lagrangian in the component fields. The bosonic part of the lagrangian reads

$$
\begin{aligned}
\mathcal{L}_{\rm kin} &= -\frac{1}{2g^2}(\partial\vec{r})^2 + \frac{g^2}{2}\Big(\partial\gamma + \sum_{a=1}^{k} A^a\Big)^2 + \\
&\quad + \sum_{a=1}^{k}\left\{\frac{1}{2e_a^2}\Big((F_{01}^a)^2 - |\partial\phi_a|^2 - |\partial\sigma_a|^2\Big) - |\mathcal{D}q_a|^2 - |\mathcal{D}\widetilde{q}_a|^2\right\} \\
\mathcal{L}_{\rm pot} &= \sum_{a=1}^{k}\Big\{ -\frac{e_a^2}{2}(|q_a|^2 - |\widetilde{q}_a|^2 - r^3 + r_a^3)^2 - \frac{e_a^2}{2}|2q_a\widetilde{q}_a - (r^1 + ir^2) + (r_a^1 + ir_a^2)|^2 - \\
&\qquad - (|\phi_a|^2 + |\sigma_a|^2)(|q_a|^2 + |\widetilde{q}_a|^2 + g^2)\Big\} \\
\mathcal{L}_{\rm top} &= -\sum_{a=1}^{k}\theta_a F_{01}^a\,, \qquad (3.5)
\end{aligned}
$$

where we denote the scalar component of P as $r^3 + ig^2\gamma$.

Let us comment on some of the properties of this lagrangian. One important property of this model is that the scalar field γ is shifted by all k U(1) gauge transformations

$$
\delta A_\mu^a = \partial_\mu\alpha_a\,, \qquad \delta\gamma = -\alpha_a \qquad (a = 1, \cdots, k)\,. \qquad (3.6)
$$

Combining γ with the phase $\varphi_a = -\arg(iq_a)$ of the hypermultiplet scalar, we can define the gauge invariant field κ as

$$
\kappa = \gamma - \sum_{a=1}^{k}\varphi_a\,. \qquad (3.7)
$$

Note that there is no dynamical theta angle θ in $\mathcal{L}_{\rm top}$ 3.5 since we have dualized the twisted chiral multiplet Θ. It is also worth mentioning that the potential term $\mathcal{L}_{\rm pot}$ is exactly the same as the H-monopole model 2.7. Therefore, the equation for the moduli space of vacua is the same as the H-monopole case 2.8.

By looking at the gauge symmetry of this model 3.4, we immediately notice that this system gives the hyperKähler quotient construction of multi-Taub-NUT space [4].[3] We can explicitly write down the low energy theory following the procedure in the previous

[3] A similar model in three dimensions was considered in [19].

section. By restricting to the vacuum 2.8 and integrating out the gauge fields, we find that A^a is given by

$$A^a_\mu = -\frac{1}{2|\vec{r}-\vec{r}_a|H}(\partial_\mu\kappa - \vec{\omega}\cdot\partial_\mu\vec{r}) - \partial_\mu\varphi_a - \frac{1}{2}\vec{\omega}_a\cdot\partial_\mu\vec{r}\,. \tag{3.8}$$

where $\vec{\omega}_a$ and $H, \vec{\omega}$ are defined in 2.11 and 2.14. Plugging this back into the lagrangian, the low energy effective lagrangian is given by

$$\mathcal{L}_{\rm eff} = \frac{1}{2}H(\partial\vec{r})^2 + \frac{1}{2H}(\partial\kappa - \vec{\omega}\cdot\partial\vec{r})^2\,. \tag{3.9}$$

This is nothing but the metric of the multi-Taub-NUT space M_k [7, 8]. As expected from the supergravity analysis of the B-field zero-mode [5], the curvature $F^a = dA^a$ of the forms 3.8 are equal to the k linearly independent L^2-normalizable self-dual harmonic two-forms on the multi-Taub-NUT space [6].[4][5]

4. The physical meaning of the collective coordinate θ_a

We have seen in section 2 that the instanton numbers in the H-monopole model correspond to the momentum modes in the target space and the collective coordinate θ_a (or theta-parameter in GLSM) is the translation mode in the $\mathbf{S}^1_\theta$. From the general property of T-duality, we expect that the instanton number in the KK-monopole model corresponds to the winding number in the target space picture and the conjugate collective coordinate θ_a is the translation mode in the "winding coordinate" $\mathbf{S}^1_{\tilde{\theta}}$.[6] The instanton corrections in the KK-monopole model is analyzed in the UV gauge theory language in [1] for the $k = 1$ case. In this section we consider the instanton effect from the low energy sigma model perspective.

As mentioned in the introduction, from the effective theory point of view it seems puzzling to have a nontrivial instanton correction for the $k = 1$ case since there is no closed two-cycle in that case. Here we propose that the relevant instanton correction is coming from a disk instanton. The possibility of disk instanton is suggested in [2] in relation to the description of KK-monopoles in the string field theory.

Since we are considering the sigma model on the sphere $\Sigma = \mathbf{S}^2$, it seems impossible to have a disk instanton. However, because of the non-compactness of the target space we can consider a map which effectively looks like a disk. We decompose the worldsheet sphere into a disk and one point

$$\mathbf{S}^2 = D^2 \cup \{\infty\} \tag{4.1}$$

[4]We make slight abuse of notation that we identify the forms on the effective target space (i.e. Taub-NUT) and their pull-back to the worldsheet.

[5]Note the difference of the L^2 harmonic two-forms on the ALF space (Taub-NUT) and the ALE space [20]

$$\mathrm{dim}L^2\mathcal{H}^2(\mathrm{ALF}_{A_{k-1}}) = k\,, \qquad \mathrm{dim}L^2\mathcal{H}^2(\mathrm{ALE}_{A_{k-1}}) = k-1\,.$$

[6]We do not try to define $\widetilde{\theta}$ as a worldsheet *field* [1]. Rather, the collective coordinates θ_a appear in the worldsheet action as *parameters*. By construction, the GLSM of H-monopoles and the GLSM of KK-monopoles have the same parameter space $(\mathbf{R}^3\times\mathbf{S}^1)^k/S_k$.

and consider a map which sends the disk into a finite region of target space and the ∞ of worldsheet to the infinity of target space. With this understanding, we can talk about the disk instanton.

Let us consider the configuration of disk instanton which wraps n times on the semi-infinite cigar in the target space

$$D^2 = \{z; |z| < 1\} \longrightarrow nC_a = \{\vec{r}(z) = \vec{r}_a + f(|z|)\vec{v},\ \kappa(z) = n\ \arg(z)\} \tag{4.2}$$

where $\vec{v}$ is a constant unit vector specifying the direction of cigar. The function $f(|z|)$ satisfies the boundary condition

$$f(0) = 0\,, \qquad f(1) = \infty\,. \tag{4.3}$$

$f(|z|)$ can be determined by minimizing the action but we don't need its explicit form. We can see that C_a has a shape of cigar by noting that $\mathbf{S}^1$ fiber is degenerate at $\vec{r} = \vec{r}_a$ and the radius of $\mathbf{S}^1$ fiber approaches a constant value g at infinity. If we regard $\tau = 1/|z|$ as the euclidean time coordinate on the worldsheet, the configuration 4.2 represents a process that a string winding n times around the $\mathbf{S}^1$ fiber at infinity propagates towards one of the monopole cores $\vec{r} = \vec{r}_a$ and unwinds at $\tau = \infty$ [2].

One can compute the flux of the harmonic two-form $F^a = dA^a$ 3.8 through the cigar C_a

$$\int_{nC_a} \frac{F^b}{2\pi} = n\delta_{a,b}\,. \tag{4.4}$$

In this computation only the first term in 3.8 contributes. Other terms in 3.8 couple to the "unwinding" string away from the core of monopoles [2]. We can understand the diagonal property 4.4 in a more abstract way. It is known that the intersection form of the harmonic two-forms $F^a = dA^a$ given by 3.8 is diagonal [6]

$$\int_{M_k} \frac{F^a}{2\pi} \wedge \frac{F^b}{2\pi} = \delta_{a,b}\,. \tag{4.5}$$

Therefor we can define C_a as a Poincare dual of the F^a

$$C_a = PD\left(\left[\frac{F^a}{2\pi}\right]\right)\,. \tag{4.6}$$

From this discussion it is clear that the instanton number in the low energy picture is the winding number of disk instanton wrapped around the cigar. The instanton action in the UV gauge theory picture is obtained in [1]

$$S^{\text{inst}} = |n||\vec{r} - \vec{r}_a| - in\theta_a\,. \tag{4.7}$$

We do not try to reproduce this instanton action from the low energy point of view. However, we note that the relevant instanton in the UV computation is the constrained instanton defined in the $g \to 0$ limit [3, 1]. In this limit the asymptotic radius g of the $\mathbf{S}^1$ fiber goes to zero. Therefore it is possible that the instaton action calculated in the low energy theory is finite despite the fact that the cigar C_a is non-compact. In general the low energy theory does not necessarily give the same instanton action as the UV theory. However we expect that the topological charge should have the same interpretation in the UV and IR. What we have shown is that the disk instanton wrapped around C_a has the same topological charge as the gauge theory instanton of $a^{\rm th}$ U(1).

When $k > 1$ one can also consider a sphere instanton which wraps the closed two-cycle in the Taub-NUT

$$\Sigma = \mathbf{S}^2 \longrightarrow C_a - C_b \,. \tag{4.8}$$

When the two KK-monopoles coincide $\vec{r}_a = \vec{r}_b$, the corresponding cycle $C_a - C_b$ vanishes. From the duality to H-monopole, we expect that the coincident KK-monopoles would show the throat behavior. It would be interesting to see this explicitly using the GLSM. It would be also interesting to determine the precise form of the instanton correction to the KK-monopole sigma model. It is observed [1] that not only the metric is corrected but also the torsion is generated. Parhaps it might be useful to consider the topological A-twist of the KK-monopole GLSM. It would be also interesting to construct a GLSM of the D_k-type Taub-NUT space [22] and a $\mathcal{N} = (4, 0)$ GLSM describing a bound state of H-monopoles and KK-monopoles.

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